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Thermodynamic restrictions, free energies and Saint-Venant's principle in the linear theory of viscoelastic materials with voids

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Abstract

This paper examines the thermodynamic restrictions imposed by the second law of thermodynamics upon the relaxation functions in the linear theory of viscoelastic materials with voids. On this basis the existence of a maximal free energy is proved by means of a constructive method. Further, we use such a maximal free energy in order to establish a principle of Saint-Venant type in the dynamics of viscoelastic materials with voids. A uniqueness theorem is proved for finite and infinite bodies and we note that it is free of any kind of a priori assumptions concerning the orders of growth of solutions at infinity.

1. Introduction

Theories of materials with fading memory have been developed to a great extent in the last three decades. In this connection we have to mention the books by Day (1972), Truesdell (1984) and Fabrizio and Morro (1992). Only recently Fabrizio et al. (1994, 1995) have given an explicit expression for the maximal Helmholtz free energy under the assumption that the constitutive equation of linear viscoelasticity obeys the requirements followed by the second law of thermodynamics.

The viscoelastic behaviour of porous solids in which the skeletal or matrix material is viscoelastic and the interstices are void of material has been studied by Cowin (1985). The linear theory of integral type for viscoelastic materials with voids was studied by Ciarletta and Scalia (1991) and some uniqueness and continuous dependence results were established. Some reciprocal and variational theorems were also established by Ciarletta (1989).

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In the present paper we consider the approach proposed by Ciarletta and Scalia (1991) for describing the viscoelastic behaviour of materials with voids. Consistent with the model of viscoelastic solid with voids, we restate the second law of thermodynamics through the work inequality and then, in the lines described by Fabrizio and Morro (1992), we derive the thermodynamic restrictions on the relaxation functions from the constitutive functionals by means of time-harmonic vibrations. Further, we use the results of Fabrizio et al. (1994, 1995) in order to establish the existence of a maximal free energy. Moreover, via a constructive method, we establish the existence of a free energy which proves to be the minimal free energy over the full history space.

On this basis we establish Saint-Venant type principle valid for the dynamic viscoelastic behaviour of materials with voids. In fact, we suppose that the viscoelastic solid with voids is subjected on the time interval $[0, T]$ to initial, body and boundary data having a bounded support $\hat{D}_T$. Then a complete description is given upon what happens outside of the support region $\hat{D}_T$. More precisely, we prove that, for each $t \in [0, T]$, there exists a bounded region $D_{ct} \supset \hat{D}_T$, so that the whole activity vanishes outside of D_{ct}; while into the region $D_{ct} \backslash \hat{D}_T$, an appropriate measure of the dynamic viscoelastic process decays spatially with the distance r from the bounded support $\hat{D}_T$, the decay rate being controlled by the factor $(1-r/ct)$, $c = \text{const}$, $c > 0$. As an immediate consequence of the Saint-Venant's principle, we get a uniqueness theorem valid for finite or infinite bodies and which is free of any kind of *a priori* assumptions concerning the behaviour of solutions at infinity.

2. Basic equations. Some preliminaries

Throughout this paper we denote $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}^{++}$, the reals, positive reals and strictly positive reals, respectively. Let $\mathscr{E}^3$ denote a real euclidean three-dimensional point space and let V be the three-dimensional vector space associated with $\mathscr{E}^3$.

We consider a body that at time $t = 0$ occupies the bounded or unbounded regular region B of euclidean three-dimensional space $\mathscr{E}^3$ and assume that its boundary ∂B is a piecewise smooth surface. We refer the motion of the body to a fixed system of rectangular Cartesian axes Ox_i $(i = 1, 2, 3)$. Let $\mathbf{n}$ be the outward unit normal of ∂B. We shall employ the usual summation and differentiation conventions: Latin subscripts (unless otherwise specified) are understood to range over the integers $(1, 2, 3)$; summation over repeated subscripts is implied and Latin subscripts preceded by a comma denote partial differential with respect to the corresponding Cartesian coordinate. Moreover, we use a superposed dot to denote partial differentiation with respect to time.

In this paper we consider the linear theory of anisotropic and inhomogeneous viscoelastic materials with voids. The equations of motion for materials with voids as described by Cowin and Nunziato (1983), Nunziato and Cowin (1979) are

$$\begin{aligned} t_{ij,j}+f_i &= \rho \ddot{u}_i, \\ h_{i,i}+g+l &= \rho\kappa\ddot{\varphi}. \end{aligned} \tag{1}$$

Here t_{ij} are the stresses, f_i is the body force per unit volume, u_i is the displacement, ρ is the density in the reference configuration, h_i is the equilibrated stress, g is the intrinsic equilibrated body force,

l is the extrinsic equilibrated body force, φ is the change in volume fraction from the reference volume fraction and κ is the equilibrated inertia. Throughout in what follows we assume that ρ and κ are continuous, bounded and strictly positive fields on the closure ($\bar{B}$) of B.

Assuming that the work done on every closed path starting from the virgin state is invariant under time reversal and following Day (1971), Ciarletta and Scalia (1991) have proposed the constitutive equations for viscoelastic materials with voids in the following form

$$t_{ij}(\mathbf{x},t) = \int_{-\infty}^{t} [G_{ijpq}(\mathbf{x},t-s)\dot{e}_{pq}(\mathbf{x},s) + B_{ij}(\mathbf{x},t-s)\dot{\varphi}(\mathbf{x},s) + D_{ijq}(\mathbf{x},t-s)\dot{\chi}_q(\mathbf{x},s)]\,\mathrm{d}s;$$

$$g(\mathbf{x},t) = -\int_{-\infty}^{t} [B_{ij}(\mathbf{x},t-s)\dot{e}_{ij}(\mathbf{x},s) + b(\mathbf{x},t-s)\dot{\varphi}(\mathbf{x},s) + D_i(\mathbf{x},t-s)\dot{\chi}_i(\mathbf{x},s)]\,\mathrm{d}s,$$

$$h_i(\mathbf{x},t) = \int_{-\infty}^{t} [D_{pqi}(\mathbf{x},t-s)\dot{e}_{pq}(\mathbf{x},s) + D_i(\mathbf{x},t-s)\dot{\varphi}(\mathbf{x},s) + A_{ij}(\mathbf{x},t-s)\dot{\chi}_j(\mathbf{x},s)]\,\mathrm{d}s, \tag{2}$$

where

$$e_{ij} \equiv \tfrac{1}{2}(u_{i,j} + u_{j,i}), \quad \chi_i \equiv \varphi_{,i}, \tag{3}$$

and $\mathbf{x} \in V$ is the position vector of a point in B. The relaxation functions G_{ijpq}, B_{ij}, D_{ijp}, D_i, b and A_{ij} have the following symmetry properties

$$G_{ijpq} = G_{pqij} = G_{jipq}, \quad A_{ij} = A_{ji}, \quad B_{ij} = B_{ji}, \quad D_{ijp} = D_{jip}, \tag{4}$$

on $\bar{B} \times [0, \infty]$. In what follows, when no confusion may occur, we suppress the dependence upon the spatial variable.

We introduce the four-dimensional linear space $\mathscr{D}_4$ as the set of all four-dimensional displacement field $\mathbf{U}$ of the form

$$\mathbf{U} \equiv \{u_i, \kappa_1\varphi\}, \quad \kappa_1 = \sqrt{\kappa}, \tag{5}$$

and define the inner product in $\mathscr{D}_4$ by

$$\mathbf{U}\cdot\mathbf{V} \equiv u_i v_i + \kappa\varphi\psi, \quad \text{for } \mathbf{U} = \{u_i, \kappa_1\varphi\}, \quad \mathbf{V} = \{v_i, \kappa_1\psi\} \in \mathscr{D}_4. \tag{6}$$

Accordingly the magnitude of the vector field $\mathbf{V} = \{v_i, \kappa_1\psi\} \in \mathscr{D}_4$ is given by

$$|\mathbf{V}| \equiv (\mathbf{V}\cdot\mathbf{V})^{1/2} = (v_i v_i + \kappa\psi^2)^{1/2}. \tag{7}$$

We denote by $\mathbf{U}^t$ the history of $\mathbf{U} \in \mathscr{D}_4$ up to time t, i.e., $\mathbf{U}^t(s) \equiv \mathbf{U}(t-s)$, $s \in \mathbb{R}^+$.

Corresponding to $\mathbf{U} = \{u_i, \kappa_1\varphi\} \in \mathscr{D}_4$, we introduce the state of strain $\mathbf{E}(\mathbf{U})$ defined by

$$\mathbf{E}(\mathbf{U}) \equiv \{e_{ij}(\mathbf{U}), \varphi, \kappa_1\chi_i(\mathbf{U})\}, \tag{8}$$

where $e_{ij}(\mathbf{U})$ and $\chi_i(\mathbf{U})$ are calculated by means of the relation (3). Further, we denote by $\mathscr{E}$ the linear space of all objects of the form (8) and define the magnitude of $\mathbf{E} \in \mathscr{E}$ by

$$|\mathbf{E}| \equiv (\mathbf{E}\cdot\mathbf{E})^{1/2} \equiv (e_{ij}e_{ij} + \varphi^2 + \kappa\chi_i\chi_i)^{1/2}. \tag{9}$$

For $\mathbf{E} \in \mathscr{E}$ we denote by $\mathbf{E}^t$ the history up to time t, i.e., $\mathbf{E}^t(s) \equiv \mathbf{E}(t-s)$, $s \in \mathbb{R}^+$. Given a history

$\mathbf{E}^t\colon \mathbb{R}^+ \to \mathscr{E}$, we denote by ${}_r\mathbf{E}^t$ the past history which is obtained by restriction of $\mathbf{E}^t$ to $\mathbb{R}^{++}$. We denote by Φ the set of admissible histories for the system under consideration and let Φ_r be the set of past histories obtained by restriction of the histories of Φ to $\mathbb{R}^{++}$.

Further, for given $\mathbf{E}^t \in \Phi$, we define $\mathbf{T}(\mathbf{E}^t)$ as follows

$$\mathbf{T}(\mathbf{E}^t) \equiv \left\{ t_{ij}(\mathbf{E}^t), \quad g(\mathbf{E}^t), \quad \frac{1}{\kappa_1} h_i(\mathbf{E}^t) \right\}, \tag{10}$$

where $t_{ij}(\mathbf{E}^t)$, $g(\mathbf{E}^t)$ and $h_i(\mathbf{E}^t)$ are calculated by means of the relation (2). According to (8) and (9), the magnitude of $\mathbf{T}(\mathbf{E}^t) = \{t_{ij}(\mathbf{E}^t), g(\mathbf{E}^t), \kappa_1[(1/\kappa)h_i(\mathbf{E}^t)]\} \in \mathscr{E}$ is given by

$$|\mathbf{T}(\mathbf{E}^t)| = \left[t_{ij}(\mathbf{E}^t) t_{ij}(\mathbf{E}^t) + g(\mathbf{E}^t)^2 + \frac{1}{\kappa} h_i(\mathbf{E}^t) h_i(\mathbf{E}^t) \right]^{1/2}. \tag{11}$$

We note that, by an appropriate integration by parts, it is possible to write the constitutive eqns (2) in the form

$$\begin{aligned} t_{ij}(\mathbf{E}^t) &= G_{ijpq}(0)e_{pq}(t) + B_{ij}(0)\varphi(t) + D_{ijq}(0)\chi_q(t) \\ &\quad + \int_0^\infty [\dot{G}_{ijpq}(s)e_{pq}(t-s) + \dot{B}_{ij}(s)\varphi(t-s) + \dot{D}_{ijq}(s)\chi_q(t-s)]\,\mathrm{d}s, \\ g(\mathbf{E}^t) &= -B_{ij}(0)e_{ij}(t) - b(0)\varphi(t) - D_i(0)\chi_i(t) \\ &\quad - \int_0^\infty [\dot{B}_{ij}(s)e_{ij}(t-s) + \dot{b}(s)\varphi(t-s) + \dot{D}_i(s)\chi_i(t-s)]\,\mathrm{d}s, \\ h_i(\mathbf{E}^t) &= D_{pqi}(0)e_{pq}(t) + D_i(0)\varphi(t) + A_{ij}(0)\chi_j(t) \\ &\quad + \int_0^\infty [\dot{D}_{pqi}(s)e_{pq}(t-s) + \dot{D}_i(s)\varphi(t-s) + \dot{A}_{ij}(s)\chi_j(t-s)]\,\mathrm{d}s. \end{aligned} \tag{12}$$

For later convenience, we introduce the following bilinear functional

$$\begin{aligned} \mathscr{F}[\mathbf{G}(s); \mathbf{E}^{(1)}, \mathbf{E}^{(2)}] \equiv \tfrac{1}{2}\{ & G_{ijpq}(s)e_{ij}^{(1)}e_{pq}^{(2)} + b(s)\varphi^{(1)}\varphi^{(2)} + A_{ij}(s)\chi_i^{(1)}\chi_j^{(2)} \\ & + D_{ijp}(s)(e_{ij}^{(1)}\chi_p^{(2)} + e_{ij}^{(2)}\chi_p^{(1)}) + B_{ij}(s)(\varphi^{(1)}e_{ij}^{(2)} + \varphi^{(2)}e_{ij}^{(1)}) \\ & + D_i(s)(\varphi^{(1)}\chi_i^{(2)} + \varphi^{(2)}\chi_i^{(1)})\}, \quad \forall \mathbf{E}^{(1)}, \mathbf{E}^{(2)} \in \mathscr{E}, \end{aligned} \tag{13}$$

where $\mathbf{G}(s) \equiv \{G_{ijpq}(s),\ b(s),\ A_{ij}(s),\ D_{ijp}(s),\ B_{ij}(s),\ D_i(s)\}$. Obviously, the symmetry relation (4) implies that we have

$$\mathscr{F}[\mathbf{G}(s); \mathbf{E}^{(1)}, \mathbf{E}^{(2)}] = \mathscr{F}[\mathbf{G}(s); \mathbf{E}^{(2)}, \mathbf{E}^{(1)}] \quad \forall \mathbf{E}^{(1)}, \mathbf{E}^{(2)} \in \mathscr{E}. \tag{14}$$

Further, we define the following quadratic form

$$\mathscr{W}(\mathbf{G}(s); \mathbf{E}) \equiv \mathscr{F}[\mathbf{G}(s); \mathbf{E}, \mathbf{E}] \quad \forall \mathbf{E} \equiv \mathscr{E}, \tag{15}$$

that is,

$$\mathscr{W}(\mathbf{G}(s);\mathbf{E}) = \tfrac{1}{2}G_{ijpq}(s)e_{ij}e_{pq}+\tfrac{1}{2}b(s)\varphi^2+\tfrac{1}{2}A_{ij}(s)\chi_i\chi_j + D_{ijp}(s)e_{ij}\chi_p+B_{ij}(s)e_{ij}\varphi+D_i(s)\chi_i\varphi, \quad \forall \mathbf{E}\in\mathscr{E}. \tag{16}$$

Finally, we recall some results on the Fourier transform as described, for example, by Baggett and Fulks (1979). For any function $f\in L^2(\mathbb{R})$ we denote by $f^{(F)}$ the Fourier transform

$$f^{(F)}(\omega) \equiv \int_{-\infty}^{\infty} f(\xi)\exp(-i\omega\xi)\,\mathrm{d}\xi. \tag{17}$$

Functions defined on $\mathbb{R}^+$ are identified with functions on $\mathbb{R}$ which vanish identically on $(-\infty,0)$. For such functions, $f^{(F)} = f^{(c)}-if^{(s)}$, where $f^{(s)}$ and $f^{(c)}$ are the Fourier sine and cosine transforms

$$f^{(s)}(\omega) \equiv \int_0^{\infty} f(\xi)\sin\omega\xi\,\mathrm{d}\xi, \quad f^{(c)}(\omega) \equiv \int_0^{\infty} f(\xi)\cos\omega\xi\,\mathrm{d}\xi. \tag{18}$$

The Plancherel's theorem for the Fourier transform gives

$$\int_{-\infty}^{\infty} f(\xi)g(\xi)\,\mathrm{d}\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} f^{(F)}(\omega)g^{(F)}{}^{*}(\omega)\,\mathrm{d}\omega, \quad \forall f,g\in L^2(\mathbb{R}), \tag{19}$$

where $*$ signifies complex conjugate.

The Fourier inversion formula gives

$$f(\xi) = \frac{2}{\pi}\int_0^{\infty} \sin\omega\xi f^{(s)}(\omega)\,\mathrm{d}\omega. \tag{20}$$

3. Thermodynamic restrictions for linear viscoelastic solids with voids

In this section we derive the thermodynamic restrictions imposed by the second law of thermodynamics on the constitutive functionals given by (12). In this aim we restate the second law of thermodynamics through the work inequality that within our context reads as

$$\int_0^d [t_{ij}(\mathbf{E}^t)\dot{e}_{ij}(t)-g(\mathbf{E}^t)\dot{\varphi}(t)+h_i(\mathbf{E}^t)\dot{\chi}_i(t)]\,\mathrm{d}t \geqslant 0 \tag{21}$$

holds for any cycle on $[0,d]$ and that equality holds if and only if $\mathbf{E}^t$ is a constant history $\mathbf{E}\dagger$ (cf. Fabrizio and Morro, 1992). If we substitute the constitutive eqns (12) in (21), we get, with (13), that

$$\int_0^d \mathscr{F}[\mathbf{G}(0);\dot{\mathbf{E}}(t),\mathbf{E}(t)]\,\mathrm{d}t+\int_0^d \mathrm{d}t\int_0^{\infty} \mathscr{F}[\dot{\mathbf{G}}(s);\dot{\mathbf{E}}(t),\mathbf{E}^t(s)]\,\mathrm{d}s \geqslant 0. \tag{22}$$

As it is well-known (see Fabrizio and Morro, 1992), the time-harmonic vibrations prove to be especially suited to the derivation of thermodynamic restrictions on the relaxation functions in viscoelasticity. Accordingly we consider oscillatory strain evolutions of the form

$$\mathbf{E}(t) = \mathbf{E}^{(1)}\cos\omega t + \mathbf{E}^{(2)}\sin\omega t, \quad \omega\in\mathbb{R}^{++}, \quad \forall\mathbf{E}^{(1)},\mathbf{E}^{(2)}\in\mathscr{E}. \tag{23}$$

Substitution into the inequality (22) with $d = 2\pi/\omega$, gives

$$\begin{aligned}\mathscr{F}[\mathbf{G}(0);\mathbf{E}^{(1)},\mathbf{E}^{(2)}] - \mathscr{F}[\mathbf{G}(0);\mathbf{E}^{(2)},\mathbf{E}^{(1)}] + \mathscr{F}[\dot{\mathbf{G}}^{(c)}(\omega);\mathbf{E}^{(1)},\mathbf{E}^{(2)}]\\ - \mathscr{F}[\dot{\mathbf{G}}^{(c)}(\omega);\mathbf{E}^{(2)},\mathbf{E}^{(1)}] + \mathscr{F}[\dot{\mathbf{G}}^{(s)}(\omega);\mathbf{E}^{(1)},\mathbf{E}^{(1)}]\\ + \mathscr{F}[\dot{\mathbf{G}}^{(s)}(\omega);\mathbf{E}^{(2)},\mathbf{E}^{(2)}] \leqslant 0, \quad \forall\mathbf{E}^{(1)},\mathbf{E}^{(2)}\in\mathscr{E}, \quad \forall\omega\in\mathbb{R}^{++},\end{aligned} \tag{24}$$

where $\dot{\mathbf{G}}^{(c)}$ and $\dot{\mathbf{G}}^{(s)}$ are the Fourier sine and cosine transforms of $\dot{\mathbf{G}}$. If we set $\omega\to\infty$ in (24), by Riemann–Lebesgue's lemma, we get

$$\mathscr{F}[\mathbf{G}(0);\mathbf{E}^{(1)},\mathbf{E}^{(2)}] = \mathscr{F}[\mathbf{G}(0);\mathbf{E}^{(2)},\mathbf{E}^{(1)}], \quad \forall\mathbf{E}^{(1)},\mathbf{E}^{(2)}\in\mathscr{E}. \tag{25}$$

Now, if we set $\omega\to 0$ in (24) and then we use (25), we get

$$\mathscr{F}[\mathbf{G}(\infty);\mathbf{E}^{(1)},\mathbf{E}^{(2)}] = \mathscr{F}[\mathbf{G}(\infty);\mathbf{E}^{(2)},\mathbf{E}^{(1)}], \quad \forall\mathbf{E}^{(1)},\mathbf{E}^{(2)}\in\mathscr{E}. \tag{26}$$

Obviously, the thermodynamic restrictions described by the relations (25) and (26) are in full accord with our assumptions concerning the symmetry of the relaxation functions presented in (4).

Further, by setting $\mathbf{E}^{(1)} = \mathbf{E}^{(2)} = \mathbf{E}$ in (24), we deduce that

$$\mathscr{F}[\dot{\mathbf{G}}^{(s)}(\omega);\mathbf{E},\mathbf{E}] \leqslant 0, \quad \forall\mathbf{E}\in\mathscr{E}, \quad \forall\omega\in\mathbb{R}^{+}, \tag{27}$$

the equality sign holds if and only if the history of $\mathbf{E}$ is constant different from zero, that is, we have $\omega = 0$. Thus, by means of the relations (15), from (27) we have

$$\mathscr{W}(\dot{\mathbf{G}}^{(s)}(\omega);\mathbf{E}) < 0, \quad \forall\omega\in\mathbb{R}^{++}, \quad \forall\mathbf{E}\in\mathscr{E}\backslash\{\mathbf{0}\} \tag{28}$$

while $\dot{\mathbf{G}}^{(s)}(0) = \mathbf{0}$.

By using the Fourier inversion formula (20), we have

$$\dot{\mathbf{G}}(\xi) = \frac{2}{\pi}\int_0^\infty \sin\omega\xi\,\dot{\mathbf{G}}^{(s)}(\omega)\,\mathrm{d}\omega, \tag{29}$$

from which, by an integration with respect to ξ, we get

$$\mathbf{G}(\xi) - \mathbf{G}(0) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos\omega\xi}{\omega}\,\dot{\mathbf{G}}^{(s)}(\omega)\,\mathrm{d}\omega. \tag{30}$$

If we use (30) in (28), we deduce

$$\mathscr{W}(\mathbf{G}(0) - \mathbf{G}(\xi);\mathbf{E}) > 0, \quad \forall\mathbf{E}\in\mathscr{E}\backslash\{\mathbf{0}\}. \tag{31}$$

Now, we set $\xi\to\infty$ in (31), in order to deduce that

$$\mathscr{W}(\mathbf{G}(0) - \mathbf{G}(\infty);\mathbf{E}) \geqslant 0, \quad \forall\mathbf{E}\in\mathscr{E}. \tag{32}$$

Further, we note that (30) gives

$$\mathbf{G}(\infty)-\mathbf{G}(0) = \frac{2}{\pi}\int_0^\infty \frac{1}{\omega}\dot{\mathbf{G}}^{(s)}(\omega)\,\mathrm{d}\omega, \tag{33}$$

and hence

$$\mathbf{G}(\xi)-\mathbf{G}(\infty) = -\frac{2}{\pi}\int_0^\infty \frac{\cos\omega\xi}{\omega}\dot{\mathbf{G}}^{(s)}(\omega)\,\mathrm{d}\omega. \tag{34}$$

Finally, we note that, by means of the relations (12), (19) and (33) we can write

$$\begin{aligned}
t_{ij}(\mathbf{E}^t) = {} & G_{ijpq}(\infty)e_{pq}(t)+B_{ij}(\infty)\varphi(t)+D_{ijq}(\infty)\chi_q(t) \\
& +\frac{2}{\pi}\int_0^\infty \left\{\dot{G}^{(s)}_{ijpq}(\omega)\left[e^{t(s)}_{pq}(\omega)-\frac{1}{\omega}e_{pq}(t)\right]+\dot{B}^{(s)}_{ij}(\omega)\left[\varphi^{t(s)}(\omega)-\frac{1}{\omega}\varphi(t)\right]\right. \\
& \left.+\dot{D}^{(s)}_{ijq}(\omega)\left[\chi^{t(s)}_q(\omega)-\frac{1}{\omega}\chi_q(t)\right]\right\}\mathrm{d}\omega, \\
g(\mathbf{E}^t) = {} & -B_{ij}(\infty)e_{ij}(t)-b(\infty)\varphi(t)-D_i(\infty)\chi_i(t) \\
& -\frac{2}{\pi}\int_0^\infty \left\{\dot{B}^{(s)}_{ij}(\omega)\left[e^{t(s)}_{ij}(\omega)-\frac{1}{\omega}e_{ij}(t)\right]+\dot{b}^{(s)}(\omega)\left[\varphi^{t(s)}(\omega)-\frac{1}{\omega}\varphi(t)\right]\right. \\
& \left.+\dot{D}^{(s)}_{i}(\omega)\left[\chi^{t(s)}_i(\omega)-\frac{1}{\omega}\chi_i(t)\right]\right\}\mathrm{d}\omega, \\
h_i(\mathbf{E}^t) = {} & D_{pqi}(\infty)e_{pq}(t)+D_i(\infty)\varphi(t)+A_{ij}(\infty)\chi_j(t) \\
& +\frac{2}{\pi}\int_0^\infty \left\{\dot{D}^{(s)}_{pqi}(\omega)\left[e^{t(s)}_{pq}(\omega)-\frac{1}{\omega}e_{pq}(t)\right]+\dot{D}^{(s)}_{i}(\omega)\left[\varphi^{t(s)}(\omega)-\frac{1}{\omega}\varphi(t)\right]\right. \\
& \left.+\dot{A}^{(s)}_{ij}(\omega)\left[\chi^{t(s)}_j(\omega)-\frac{1}{\omega}\chi_j(t)\right]\right\}\mathrm{d}\omega.
\end{aligned} \tag{35}$$

Throughout the remainder of this paper we assume that the relaxation function $\mathbf{G}(s)$ satisfies the thermodynamic restrictions described by the relations (25), (26), (28) and (32). Furthermore, we complete the above thermodynamic restrictions by assuming that $\mathbf{G}(s)$ satisfies the symmetry relation (4) and that it is continuous and bounded on $\bar{B}$ for each $s\in[0,\infty)$. Moreover, we assume that the quadratic form $\mathscr{W}(\mathbf{G}(\infty);\mathbf{E})$ is positive definite. As a consequence (see also Mehrabadi et al. (1993), for the problem of lower and upper bounds for the elastic strain energy), it follows that there exists the positive constant $\mu_M>0$ so that

$$2\mathscr{W}(\mathbf{G}(\infty);\mathbf{E}) \leqslant \mu_M|\mathbf{E}|^2, \quad \forall\mathbf{E}\in\mathscr{E}. \tag{36}$$

Further, the thermodynamic restriction described by the relation (32) implies that the quadratic form $\mathscr{W}(\mathbf{G}(0)-\mathbf{G}(\infty);\mathbf{E})$ is positive and, therefore, by a similar argument as before, it follows that there exists the positive constant $v_M>0$ so that

$$2\mathscr{W}(\mathbf{G}(0)-\mathbf{G}(\infty);\mathbf{E}) \leqslant \nu_M|\mathbf{E}|^2, \quad \forall \mathbf{E}\in\mathscr{E}. \tag{37}$$

Finally, we note that by means of the Cauchy–Schwarz inequality, from the relations (13) and (15) and the positiveness of $\mathscr{W}(\mathbf{G}(\infty);\mathbf{E})$, we have

$$\begin{aligned}2\mathscr{F}[\mathbf{G}(\infty);\mathbf{E}^{(1)},\mathbf{E}^{(2)}] &\leqslant [2\mathscr{W}(\mathbf{G}(\infty);\mathbf{E}^{(1)})]^{1/2}[2\mathscr{W}(\mathbf{G}(\infty);\mathbf{E}^{(2)})]^{1/2}\\ &\leqslant \mu_M^{1/2}|\mathbf{E}^{(2)}|[2\mathscr{W}(\mathbf{G}(\infty);\mathbf{E}^{(1)}]^{1/2}, \quad \forall \mathbf{E}^{(1)},\mathbf{E}^{(2)}\in\mathscr{E}.\end{aligned} \tag{38}$$

4. Maximal free energy

According with the lines described by Fabrizio et al. (1995) for linear viscoelasticity, here we will say that a functional $\psi(\dot{\mathbf{E}}(t),_r\mathbf{E}^t)$ on $\Phi = \mathscr{E}\times\Phi_r$ is a free energy if

(i) ψ is continuous on $\mathscr{E}\times\Phi_r$, differentiable with respect to the first argument, and

$$\begin{aligned}t_{ij}(\mathbf{E}^t) &= \frac{\partial\psi}{\partial e_{ij}}(\mathbf{E}(t),_r\mathbf{E}^t),\\ g(\mathbf{E}^t) &= -\frac{\partial\psi}{\partial\varphi}(\mathbf{E}(t),_r\mathbf{E}^t),\\ h_i(\mathbf{E}^t) &= \frac{\partial\psi}{\partial\chi_i}(\mathbf{E}(t),_r\mathbf{E}^t);\end{aligned} \tag{39}$$

(ii) for each value of τ such that $\dot{\mathbf{E}}(t+\tau)$ is continuous, ψ satisfies the inequality

$$\dot{\psi}(E^{t+\tau}) \leqslant t_{ij}(\mathbf{E}^{t+\tau})\dot{e}_{ij}(t+\tau)-g(\mathbf{E}^{t+\tau})\dot{\varphi}(t+\tau)+h_i(\mathbf{E}^{t+\tau})\dot{\chi}_i(t+\tau); \tag{40}$$

(iii) the functional ψ is minimal in correspondence with the constant histories in that

$$\psi(\mathbf{E}^t) \geqslant \psi(\mathbf{E}\dagger), \quad \forall \mathbf{E}^t\in\Phi, \tag{41}$$

and equality holds if and only if $\mathbf{E}^t$ is a constant history, $\mathbf{E}^t = \mathbf{E}\dagger$.

The maximal free energy ψ_M is a free energy for which we have

$$\dot{\psi}_M(\mathbf{E}^t) = t_{ij}(\mathbf{E}^t)\dot{e}_{ij}(t)-g(\mathbf{E}^t)\dot{\varphi}(t)+h_i(\mathbf{E}^t)\dot{\chi}_i(t). \tag{42}$$

We note that in the relations (39), (40) and (42), the functionals for $t_{ij}(\mathbf{E}^t)$, $g(\mathbf{E}^t)$ and $h_i(\mathbf{E}^t)$ are given by the relation (2) or, equivalently, by the relation (35).

Consider the functional

$$\begin{aligned}\psi_M(\mathbf{E}^t(t),_r\mathbf{E}^t) = \mathscr{W}(\mathbf{G}(\infty);\mathbf{E}(t)) - \frac{2}{\pi}\int_0^\infty \Bigg\{\omega\mathscr{W}\left(\dot{\mathbf{G}}^{(s)}(\omega);\left[\mathbf{E}^{t(s)}(\omega)-\frac{1}{\omega}\mathbf{E}(t)\right]\right)\\ +\omega\mathscr{W}(\dot{\mathbf{G}}^{(s)}(\omega);\mathbf{E}^{t(c)}(\omega))\Bigg\}\mathrm{d}\omega, \quad \forall(\mathbf{E}(t),_r\mathbf{E}^t)\in\mathscr{E}\times\Phi_r,\end{aligned} \tag{43}$$

where $\mathbf{E}^{t(s)}$ and $\mathbf{E}^{t(c)}$ are the Fourier sine and cosine transforms of $\mathbf{E}^t$.

We first note that, in view of the positiveness of $\mathscr{W}(\mathbf{G}(\infty);\mathbf{E}(t))$ and by using the thermodynamic restriction (28), it follows that the functional ψ_M given by the relation (43) defines a norm on Φ and the completion of Φ relative to this norm is a Banach space $\mathscr{H}_M$.

On the other hand, by using the relation (16), we see that the partial derivative of ψ_M with respect to e_{ij} is

$$\frac{\partial\psi_M}{\partial e_{ij}}(\mathbf{E}^t) = G_{ijpq}(\infty)e_{pq}(t)+D_{ijp}(\infty)\chi_p(t)+B_{ij}(\infty)\varphi(t)$$
$$+\frac{2}{\pi}\int_0^\infty\left\{\dot{G}^{(s)}_{ijpq}(\omega)\left[e^{t(s)}_{pq}(\omega)-\frac{1}{\omega}e_{pq}(t)\right]+\dot{D}^{(s)}_{ijp}(\omega)\left[\chi^{t(s)}_p(\omega)-\frac{1}{\omega}\chi_p(t)\right]\right.$$
$$\left.+\dot{B}^{(s)}_{ij}(\omega)\left[\varphi^{t(s)}(\omega)-\frac{1}{\omega}\varphi(t)\right]\right\}\mathrm{d}\omega, \quad (44)$$

that is, by means of the relation $(35)_1$, it follows that the relation $(39)_1$ holds true. By a similar procedure are obtained the other two relations in (39).

Further, by differentiating with respect to time, we have

$$\dot{\psi}_M(\mathbf{E}^t) = 2\mathscr{F}[\mathbf{G}(\infty);\mathbf{E}(t),\dot{\mathbf{E}}(t)]$$
$$-\frac{4}{\pi}\int_0^\infty\left\{\omega\mathscr{F}\left[\dot{\mathbf{G}}^{(s)}(\omega);\left(\mathbf{E}^{t(s)}(\omega)-\frac{1}{\omega}\mathbf{E}(t)\right),\left(\dot{\mathbf{E}}^{t(s)}(\omega)-\frac{1}{\omega}\dot{\mathbf{E}}(t)\right)\right]\right.$$
$$\left.+\omega\mathscr{F}[\dot{\mathbf{G}}^{(s)}(\omega);\mathbf{E}^{t(c)}(\omega),\dot{\mathbf{E}}^{t(c)}(\omega)]\right\}\mathrm{d}\omega. \quad (45)$$

If we use the relation (18) and an appropriate integration by parts, we get

$$\dot{\mathbf{E}}^{t(s)}(\omega) = \omega\mathbf{E}^{t(c)}(\omega), \quad \dot{\mathbf{E}}^{t(c)}(\omega) = \mathbf{E}(t)-\omega\mathbf{E}^{t(s)}(\omega), \quad (46)$$

so that the relation (45) becomes

$$\dot{\psi}_M(\mathbf{E}^t) = 2\mathscr{F}[\mathbf{G}(\infty);\mathbf{E}(t),\dot{\mathbf{E}}(t)]+\frac{4}{\pi}\int_0^\infty\mathscr{F}\left[\dot{\mathbf{G}}^{(s)}(\omega);\left(\mathbf{E}^{t(s)}(\omega)-\frac{1}{\omega}\mathbf{E}(t)\right),\dot{\mathbf{E}}(t)\right]\mathrm{d}\omega. \quad (47)$$

According to notation given in the relation (13) and by using the constitutive equations in the form (35), we may conclude from (47) that the relation (42) holds true.

Finally, for constant histories $\mathbf{E}^\dagger(s) = \mathbf{E}(t)$, by (18) we have that

$$\omega\mathbf{E}^{\dagger(s)}(\omega) = \mathbf{E}(t), \quad \omega\mathbf{E}^{\dagger(c)}(\omega) = \mathbf{0}, \quad (48)$$

and so, from (43), we have

$$\psi_M(\mathbf{E}^\dagger) = \mathscr{W}(\mathbf{G}(\infty);\mathbf{E}(t)). \quad (49)$$

Since the integral in (43) is a positive definite quadratic functional, from the relations (43) and (49), we deduce that the relation (41) holds true and, therefore, the functional ψ_M as defined by (43) is a maximal free energy.

In what follows we use the expression (43) for the maximal free energy in order to get an estimate for the magnitude of the stress tensor and the equilibrated stress. In this aim we note that the relations (11), (13) and (35), give

$$|\mathbf{T}(\mathbf{E}^t)|^2 = 2\mathscr{F}[\mathbf{G}(\infty);\mathbf{E}(t),\mathbf{T}^*(\mathbf{E}^t)]+\frac{4}{\pi}\int_0^\infty \mathscr{F}\left[\dot{\mathbf{G}}^{(s)}(\omega);\left(\mathbf{E}^{t(s)}(\omega)-\frac{1}{\omega}\mathbf{E}(t)\right),\mathbf{T}^*(\mathbf{E}^t)\right]\mathrm{d}\omega, \tag{50}$$

where $\mathbf{T}^*(\mathbf{E}^t) = \{t_{ij}(\mathbf{E}^t),\ -g(\mathbf{E}^t),\ \kappa_1[(1/\kappa)h_i(\mathbf{E}^t)]\}\in\mathscr{E}$.

On the basis of the Cauchy–Schwarz inequality and by using the relations (15), (28), (33), (36)–(38), from (50) we deduce

$$\begin{aligned}|\mathbf{T}(\mathbf{E}^t)|^2 &\leqslant [2\mathscr{W}(\mathbf{G}(\infty);\mathbf{T}^*(\mathbf{E}^t)]^{1/2}[2\mathscr{W}(\mathbf{G}(\infty);\mathbf{E}(t))]^{1/2}\\ &\quad+\left[-\frac{4}{\pi}\int_0^\infty \omega\mathscr{W}\left(\dot{\mathbf{G}}^{(s)}(\omega);\left(\mathbf{E}^{t(s)}(\omega)-\frac{1}{\omega}\mathbf{E}(t)\right)\right)\mathrm{d}\omega\right]^{1/2}\\ &\quad\cdot\left[-\frac{4}{\pi}\int_0^\infty \frac{1}{\omega}\mathscr{W}(\dot{\mathbf{G}}^{(s)}(\omega);\mathbf{T}^*(\mathbf{E}^t))\,\mathrm{d}\omega\right]^{1/2}\leqslant \mu_M^{1/2}|\mathbf{T}(\mathbf{E}^t)|[2\mathscr{W}(\mathbf{G}(\infty);\mathbf{E}(t))]^{1/2}\\ &\quad+\nu_M^{1/2}|\mathbf{T}(\mathbf{E}^t)|\left[-\frac{4}{\pi}\int_0^\infty \omega\mathscr{W}\left((\dot{\mathbf{G}}^{(s)}(\omega);\left(\mathbf{E}^{t(s)}(\omega)-\frac{1}{\omega}\mathbf{E}(t)\right)\right)\mathrm{d}\omega\right]^{1/2}.\end{aligned} \tag{51}$$

Thus, from (43) and (51), we deduce

$$|\mathbf{T}(\mathbf{E}^t)|^2 \leqslant 2c_0\psi_M(\mathbf{E}^t),\quad \forall\mathbf{E}^t\in\Phi, \tag{52}$$

where

$$c_0 = 2\max\{\mu_M,\nu_M\}. \tag{53}$$

Finally, we remark that, from the relations (11) and (52), we have

$$\left[t_{ij}(\mathbf{E}^t)t_{ij}(\mathbf{E}^t)+\frac{1}{\kappa}h_i(\mathbf{E}^t)h_i(\mathbf{E}^t)\right]\leqslant 2c_0\psi_M(\mathbf{E}^t),\quad \forall\mathbf{E}^t\in\Phi. \tag{54}$$

5. Saint-Venant's principle

In this section we consider a motion of a solid with voids described by the displacement $\mathbf{U}\in\mathscr{D}_4$ and generated by the action of the body force f_i and the extrinsic equilibrated body force l and by the initial and boundary data. The components of the surface traction t_i and equilibrated surface traction h at regular points of ∂B are defined by

$$t_i = t_{ij}n_j,\quad h = h_in_i, \tag{55}$$

respectively.

In what follows we consider a prescribed time interval $[0, T]$ and denote by $\hat{D}_T$ the set of all points of $\bar{B}$ such that:

(i) if $\mathbf{x} \in B$, then

$$u_i(\mathbf{x}, \tau) \neq 0 \quad \text{or } \dot{u}_i(\mathbf{x}, \tau) \neq 0 \quad \text{or } \varphi(\mathbf{x}, \tau) \neq 0 \quad \text{or } \dot{\varphi}(\mathbf{x}, \tau) \neq 0 \quad \text{for some } \tau \in (-\infty, 0], \tag{56}$$

or

$$f_i(\mathbf{x}, \tau) \neq 0 \quad \text{or } l(\mathbf{x}, \tau) \neq 0 \quad \text{for some } \tau \in [0, T]; \tag{57}$$

(ii) if $\mathbf{x} \in \partial B$, then

$$t_i(\mathbf{x}, \tau)\dot{u}_i(\mathbf{x}, \tau) \neq 0 \quad \text{or } h(\mathbf{x}, \tau)\dot{\varphi}(\mathbf{x}, \tau) \neq 0 \quad \text{for some } \tau \in [0, T]. \tag{58}$$

We note that the set $\hat{D}_T$ represents the support of the body force and of the extrinsic equilibrated body force and of the initial and boundary data. Throughout this section we assume that $\hat{D}_T$ is a bounded region.

Further, we consider a non-empty bounded regular region $\hat{D}_T^*$ which is such that $\hat{D}_T \subset \hat{D}_T^* \subset \bar{B}$. We note that if the support of data is non-empty then we choose $\hat{D}_T^*$ to be the smallest regular region in $\bar{B}$ which includes $\hat{D}_T$; in particular, $\hat{D}_T^*$ is chosen to be $\hat{D}_T$ if $\hat{D}_T$ also happens to be a regular region. If $\hat{D}_T = \varnothing$ then $\hat{D}_T^*$ may be chosen in an arbitrary manner.

We define the set D_r by

$$D_r \equiv \{\mathbf{x} \in \bar{B} \colon \hat{D}_T^* \cap \overline{\Sigma(\mathbf{x}, r)} \neq \varnothing\} \tag{59}$$

where $\Sigma(\mathbf{x}, r)$ is the open ball with radius r and the center at $\mathbf{x}$. We denote by S_r the portion of the boundary of D_r which is contained in the inside of B. We also introduce the notation $B_r \equiv B \backslash D_r$.

By using the results described by Chiritá and Quintanilla (1996) and Chiritá et al. (1996), in this section we establish a Saint-Venant's principle valid for viscoelastic materials with voids. Proceeding to make this we first observe that, as a consequence of the above assumptions, S_r is a closed regular surface. Thus, associated with $\mathbf{U} \in \mathscr{D}_4$, we can introduce the following auxiliary function

$$I(r, t) \equiv -\int_0^t \int_{S_r} [t_i(\mathbf{U})\dot{u}_i + h(\mathbf{U})\dot{\varphi}]\, \mathrm{d}A\, \mathrm{d}s, \quad r \geqslant 0, \quad 0 \leqslant t \leqslant T, \tag{60}$$

where $t_i(\mathbf{U})$ and $h(\mathbf{U})$ are defined by the relation (55).

By using the relations (56)–(58) describing the definition of $\hat{D}_T$, from the relations (55) and (60) and by means of the divergence theorem and the relations (1), (42) and (43), it follows that

$$I(r, t) - I(\hat{r}, t) = -\int_{B(\hat{r}, r)} \left[\frac{1}{2}\rho(\dot{u}_i\dot{u}_i + \kappa\dot{\varphi}^2) + \psi_M\right] \mathrm{d}V, \quad 0 \leqslant \hat{r} \leqslant r, \quad 0 \leqslant t \leqslant T, \tag{61}$$

where $B(\hat{r}, r) \equiv B_{\hat{r}} \backslash B_r$.

Further, we note that the relation (60) implies

$$\frac{\partial I}{\partial t}(r,t) = -\int_{S_r} [t_{ij}(\mathbf{U})n_j\dot{u}_i + h_i(\mathbf{U})n_i\dot{\varphi}]\,\mathrm{d}A, \tag{62}$$

while the relation (61) gives

$$\frac{\partial I}{\partial r}(r,t) = -\int_{s_r} \left[\frac{1}{2}\rho(\dot{u}_i\dot{u}_i + \kappa\dot{\varphi}^2) + \psi_M\right]\mathrm{d}A. \tag{63}$$

Using the arithmetic-geometric mean inequality and the relation (54), from the relation (62) we deduce that

$$\left|\frac{\partial I}{\partial r}(r,t)\right| \leqslant c\int_{S_r} \left[\frac{1}{2}\rho(\dot{u}_i\dot{u}_i + \kappa\dot{\varphi}^2) + \psi_M\right]\mathrm{d}A \tag{64}$$

where

$$c = \sqrt{\frac{c_0}{\rho_0}}, \quad \rho_0 = \inf\{\rho(\mathbf{x}) : \mathbf{x} \in \bar{B}\}. \tag{65}$$

On combining the relations (63) and (64) we arrive at the differential inequality

$$\left|\frac{\partial I}{\partial t}(r,t)\right| + c\frac{\partial I}{\partial r}(r,t) \leqslant 0, \quad \forall r \geqslant 0, \quad t \in [0,T], \tag{66}$$

which is equivalent to the following pair of inequalities

$$\frac{\partial I}{\partial t}(r,t) + c\frac{\partial I}{\partial r}(r,t) \leqslant 0, \quad r \geqslant 0, \quad t \in [0,T], \tag{67}$$

$$-\frac{\partial I}{\partial t}(r,t) + c\frac{\partial I}{\partial r}(r,t) \leqslant 0, \quad r \geqslant 0, \quad t \in [0,T]. \tag{68}$$

Let us suppose that B is a bounded regular region. Then by using a procedure similar to that used for obtaining the relation (61), we obtain

$$I(r,t) = U(r,t), \tag{69}$$

where

$$U(r,t) = \int_{B_r} \left[\frac{1}{2}\rho(\dot{u}_i\dot{u}_i + \kappa\dot{\varphi}^2) + \psi_M\right]\mathrm{d}V. \tag{70}$$

Let us now suppose that B is an unbounded regular region, that is $r \in [0,\infty)$. In this case the inequality (67) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\{I[r_0 + c(t - t_0), t]\} \leqslant 0, \quad \forall t \in [0,T], \tag{71}$$

while the relation (68) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\{I[r_0-c(t-t_0),t]\} \geqslant 0, \quad \forall t \in [0,T], \tag{72}$$

where (r_0, t_0) denotes an appropriate point in the (r, t)-plane so that $r_0 \geqslant 0$ and $t_0 \in [0, T]$. If we choose $r_0 \geqslant 0$ so that $r_0 - ct_0 \geqslant 0$, then, from the relations (71) and (72), we infer

$$I(r_0, t_0) \leqslant I(r_0 - ct_0, 0) = 0, \tag{73}$$

and

$$I(r_0, t_0) \geqslant I(r_0 + ct_0, 0) = 0, \tag{74}$$

respectively. If we make $r_0 \to \infty$, with $r_0 \geqslant ct_0$, from the relations (73) and (74), we deduce

$$\lim_{r\to\infty} I(r,t) = 0, \quad \forall t \in [0,T]. \tag{75}$$

On letting $r \to \infty$ in (61) and by using the relations (70) and (75), we obtain

$$I(\hat{r},t) = \int_{B(\hat{r},\infty)} \left[\frac{1}{2}\rho(\dot{u}_i\dot{u}_i + \kappa\dot{\phi}^2) + \psi_M\right]\mathrm{d}V = U(\hat{r},t), \tag{76}$$

which shows that $U(r, t)$ is well-defined for unbounded regions, and that the relation (69) remains valid for the case of an unbounded regular region.

If we substitute the relation (69) in (67), we deduce that

$$\frac{\partial U}{\partial t}(r,t) + c\frac{\partial U}{\partial r}(r,t) \leqslant 0, \quad r \geqslant 0, \quad t \in [0,T], \tag{77}$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}[U(ct,t)] \leqslant 0, \quad \forall t \in [0,T], \tag{78}$$

and consequently

$$U(ct,t) \leqslant U(0,0) = 0, \quad \forall t \in [0,T]. \tag{79}$$

From the relations (63) and (69), we deduce that, for each fixed $t \in [0, T]$, $U(r, t)$ is a non-increasing function of r, so that

$$U(r,t) \leqslant U(ct,t), \quad \text{for } r \geqslant ct, \quad t \in [0,T]. \tag{80}$$

Thus, from the relations (79) and (80), we deduce that

$$U(r,t) = 0 \quad \text{for } r \geqslant ct, t \in [0,T]. \tag{81}$$

Let us now consider the case $r \leqslant ct$, $t \in [0, T]$. We introduce the following measure

$$U^*(r,t) = \int_0^t U(r,\tau)\,\mathrm{d}\tau, \quad t \in [0,T], \quad r \geqslant 0. \tag{82}$$

If we take $r_1 \leqslant ct_1$, $t_1 \in [0, T]$, then, from the relations (81) and (82), we have

$$U^*(r_1, t_1) = \int_{r_1/c}^{t_1} U(r_1, \tau)\,\mathrm{d}\tau, \quad r_1 \geqslant 0. \tag{83}$$

Further, we use the changement of variable

$$\tau = \left(1 - \frac{r_1}{ct_1}\right)s + \frac{r_1}{c}, \tag{84}$$

in the integral of the relation (83), so that we get

$$U^*(r_1, t_1) = \left(1 - \frac{r_1}{ct_1}\right)\int_0^{t_1} U\left(r_1, \left(1 - \frac{r_1}{ct_1}\right)s + \frac{r_1}{c}\right)\mathrm{d}s. \tag{85}$$

From the relation (84) it results that

$$s \leqslant \tau, \tag{86}$$

and so, from the relations (69) and (71), with $r_0 = r_1$ and $t_0 = \tau$, it follows that

$$U(r_1, \tau) \leqslant U(r_1 + c(s - \tau), s) = U\left(\frac{r_1 s}{t_1}, s\right). \tag{87}$$

Since $U(r, t)$ is a non-increasing function of r, for all fixed $t \in [0, T]$, we infer

$$U\left(\frac{r_1 s}{t_1}, s\right) \leqslant U(0, s), \quad s \in [0, t_1]. \tag{88}$$

Thus, from (87) and (88), we deduce that

$$U\left(r_1, \left(1 - \frac{r_1}{ct_1}\right)s + \frac{r_1}{c}\right) \leqslant U(0, s), \quad s \in [0, t_1], \tag{89}$$

and so we obtain from (85) that

$$U^*(r_1, t_1) \leqslant \left(1 - \frac{r_1}{ct_1}\right)U^*(0, t_1), \quad \text{for } r_1 \leqslant ct_1,\ t_1 \in [0, T]. \tag{90}$$

Summarizing the results (81) and (90), we see that we have established the following Saint-Venant's principle:

(i) for $r \geqslant ct$, $t \in [0, T]$,

$$U(r, t) = 0; \tag{91}$$

(ii) for $r \leqslant ct$, $t \in [0, T]$,

$$U^*(r,t) \leqslant \left(1 - \frac{2}{ct}\right) U^*(0,t). \tag{92}$$

6. Discussion and conclusions

It can be seen that our Saint-Venant's principle implies an improved result of so-called domain of influence theorem. In fact, the relation (91) implies that

$$u_i(\mathbf{x},t) = 0, \quad \varphi(\mathbf{x},t) = 0, \quad \text{for}\, r \geqslant ct, \quad \forall t \in [0,T] \tag{93}$$

which implies, in particular, that

$$u_i(\mathbf{x},T) = 0, \quad \varphi(\mathbf{x},T) = 0, \quad \text{for}\, r \geqslant cT, \tag{94}$$

a result known as the domain of influence theorem (see, for example, Gurtin, 1972; Carbonaro and Russo, 1984).

We further remark that if B is a bounded regular region, then for values of T sufficiently large so as there exists a value of $t \in [0; T]$ having the property that $D_{ct} = B$, the relation (91) becomes superfluous and the behaviour of solutions is described entirely by the relation (92). On the other hand, for values of T sufficiently small, the behaviour of solutions will be described by the relation (91) almost throughout in B. Similar arguments are valid for an unbounded regular region. Concluding, we can say that our results (91) and (92) in the Saint-Venant's principle, are both pertinent even in the presence of a domain of influence theorem discussed in the above.

The Saint-Venant's principle described by the relations (91) and (92) implies a uniqueness theorem for the solutions of the initial boundary value problems associated with the model of viscoelastic materials with voids. In fact, if we consider the difference between two solutions having the same body loadings and the same initial and boundary data, then for the difference, $\hat{D}_T$ is empty for each $T \in (0, \infty)$ and so $\hat{D}_T^*$ can be chosen in an arbitrary manner. Thus, $U^*(0,t) = 0$ and, therefore, the relations (91) and (92) imply that the two solutions coincide on $(\bar{B} \backslash \hat{D}_T^*) \times (-\infty, \infty)$. Since $\hat{D}_T^*$ was arbitrarily chosen, it follows that the two solutions coincide on $\bar{B} \times (-\infty, \infty)$ and we have a uniqueness theorem. It is worth remarking that such a uniqueness result is valid for bounded as well as unbounded regular regions. It is established without any kind of artificial *a priori* assumptions on the order of growth of solutions at infinity.

We conclude this section by summarizing that we have established the Saint-Venant's principle described by the relations (91) and (92) and the uniqueness results by using a minimal set of thermodynamic restrictions imposed by the second law of thermodynamics on the relaxation functions which appear in the constitutive equations (2). Thus, we have used the mildest restrictions on the relaxation functions in order to assure that ψ_M, as given by the relation (43), to be a maximal free energy and so this can define a norm on the space of history $\mathscr{H}_M$. We have to mention that our analysis presented in the above, remains valid when ψ_M is substituted by any other free energy ψ. However, the restrictions on the relaxation functions in order to assure that ψ is a free energy are more restrictive. As in linear viscoelasticity (see Chiritá et al., 1996), it can be seen that the Banach space $\mathscr{H}$ obtained by the completion of Φ in the norm induced by the free energy ψ is larger than $\mathscr{H}_M$ because it includes, for example, bounded periodic histories which are not in $\mathscr{H}_M$.

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